

# 3.2 BINARY SEARCH TREES

- ▶ BSTs
- ordered operations
- deletion

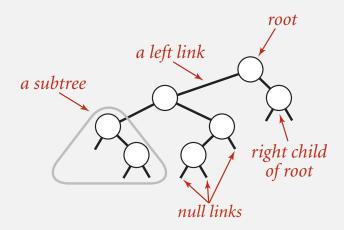
# 3.2 BINARY SEARCH TREES ▶ BSTs ordered operations deletion Algorithms ROBERT SEDGEWICK | KEVIN WAYNE http://algs4.cs.princeton.edu

### Binary search trees

Definition. A BST is a binary tree in symmetric order.

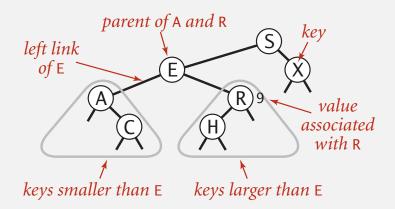
#### A binary tree is either:

- Empty.
- Two disjoint binary trees (left and right).



Symmetric order. Each node has a key, and every node's key is:

- Larger than all keys in its left subtree.
- Smaller than all keys in its right subtree.



#### **BST** representation in Java

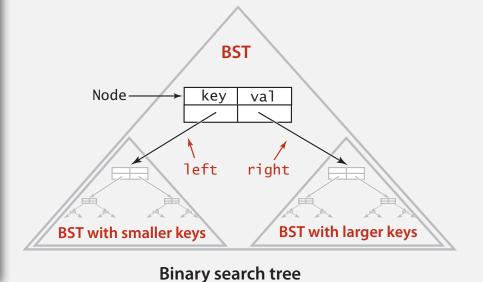
Java definition. A BST is a reference to a root Node.

A Node is comprised of four fields:

- A Key and a Value.
- A reference to the left and right subtree.

```
smaller keys larger keys
```

```
private class Node
{
   private Key key;
   private Value val;
   private Node left, right;
   public Node(Key key, Value val)
   {
      this.key = key;
      this.val = val;
   }
}
```



Key and Value are generic types; Key is Comparable

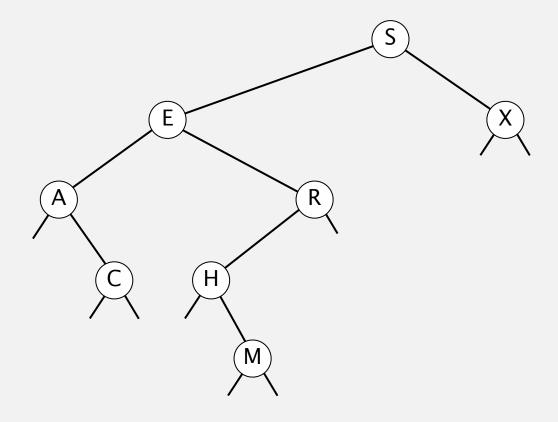
#### BST implementation (skeleton)

```
public class BST<Key extends Comparable<Key>, Value>
                                                            root of BST
    private Node root;
   private class Node
   { /* see previous slide */ }
   public void put(Key key, Value val)
   { /* see next slides */ }
   public Value get(Key key)
   { /* see next slides */ }
   public void delete(Key key)
   { /* see next slides */ }
   public Iterable<Key> iterator()
   { /* see next slides */ }
}
```

## Binary search tree demo

Search. If less, go left; if greater, go right; if equal, search hit.

#### successful search for H

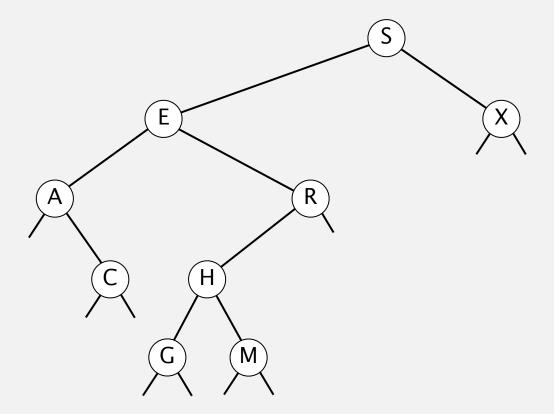




## Binary search tree demo

Insert. If less, go left; if greater, go right; if null, insert.

#### insert G



#### BST search: Java implementation

Get. Return value corresponding to given key, or null if no such key.

```
public Value get(Key key)
{
   Node x = root;
   while (x != null)
   {
      int cmp = key.compareTo(x.key);
      if (cmp < 0) x = x.left;
      else if (cmp > 0) x = x.right;
      else if (cmp == 0) return x.val;
   }
   return null;
}
```

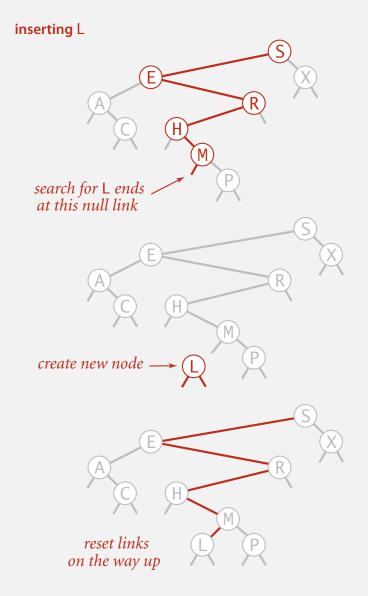
Cost. Number of compares is equal to 1 + depth of node.

#### **BST** insert

Put. Associate value with key.

Search for key, then two cases:

- Key in tree ⇒ reset value.
- Key not in tree ⇒ add new node.



Insertion into a BST

#### BST insert: Java implementation

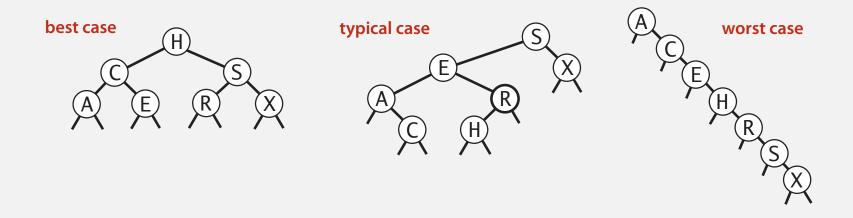
Put. Associate value with key.

```
concise, but tricky,
                                           recursive code:
public void put(Key key, Value val)
                                            read carefully!
{ root = put(root, key, val); }
private Node put(Node x, Key key, Value val)
  if (x == null) return new Node(key, val);
   int cmp = key.compareTo(x.key);
   if (cmp < 0)
      x.left = put(x.left, key, val);
   else if (cmp > 0)
      x.right = put(x.right, key, val);
   else if (cmp == 0)
      x.val = val;
   return x;
```

Cost. Number of compares is equal to 1 + depth of node.

## Tree shape

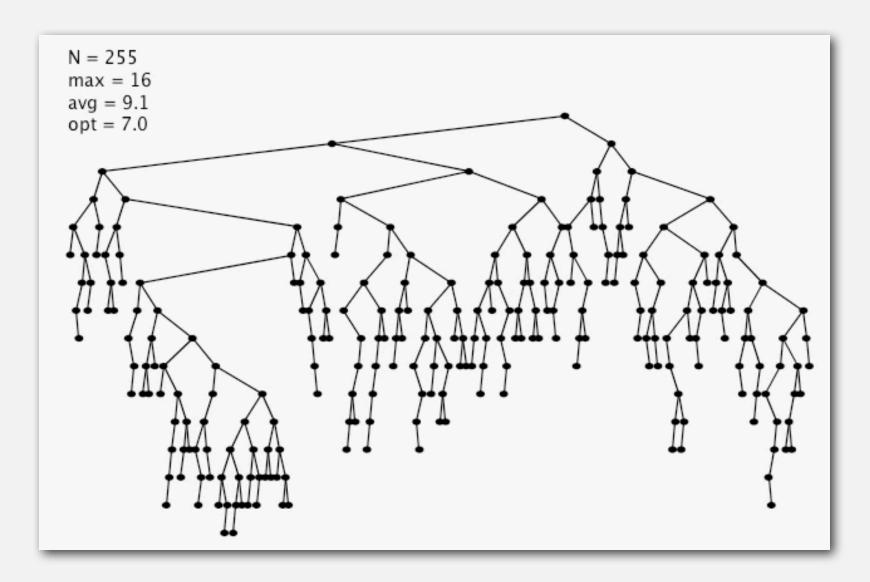
- Many BSTs correspond to same set of keys.
- Number of compares for search/insert is equal to 1 + depth of node.



Remark. Tree shape depends on order of insertion.

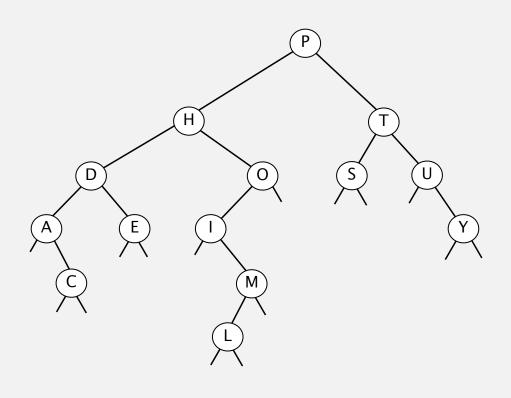
## BST insertion: random order visualization

Ex. Insert keys in random order.



## Correspondence between BSTs and quicksort partitioning





Remark. Correspondence is 1-1 if array has no duplicate keys.

#### BSTs: mathematical analysis

Proposition. If N distinct keys are inserted into a BST in random order, the expected number of compares for a search/insert is  $\sim 2 \ln N$ . Pf. 1-1 correspondence with quicksort partitioning.

Proposition. [Reed, 2003] If N distinct keys are inserted in random order,

expected height of tree is  $\sim 4.311 \ln N$ .

#### How Tall is a Tree?

Bruce Reed CNRS, Paris, France reed@moka.ccr.jussieu.fr

#### ABSTRACT

Let  $H_n$  be the height of a random binary search tree on n nodes. We show that there exists constants  $\alpha = 4.31107...$  and  $\beta = 1.95...$  such that  $\mathbf{E}(H_n) = \alpha \log n - \beta \log \log n + O(1)$ , We also show that  $\operatorname{Var}(H_n) = O(1)$ .

But... Worst-case height is *N*.

(exponentially small chance when keys are inserted in random order)

## ST implementations: summary

implementation	guarantee		average case		ordered	operations
	search	insert	search hit	insert	ops?	on keys
sequential search (unordered list)	N	N	N/2	N	no	equals()
binary search (ordered array)	lg N	N	lg N	N/2	yes	compareTo()
BST	N	N	1.39 lg N	1.39 lg N	next	compareTo()

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# 3.2 BINARY SEARCH TREES

BSTs

ordered operations

deletion

Algorithms

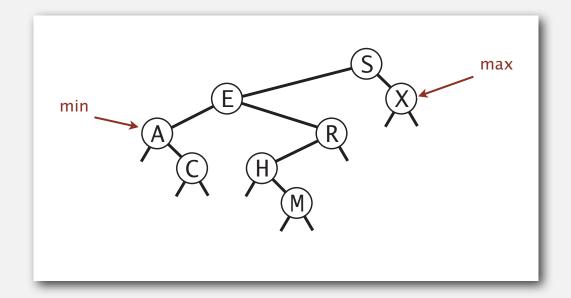
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## Minimum and maximum

Minimum. Smallest key in table.

Maximum. Largest key in table.

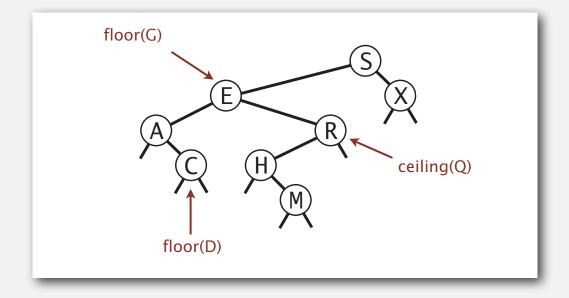


Q. How to find the min / max?

## Floor and ceiling

Floor. Largest key  $\leq$  a given key.

Ceiling. Smallest key  $\geq$  a given key.



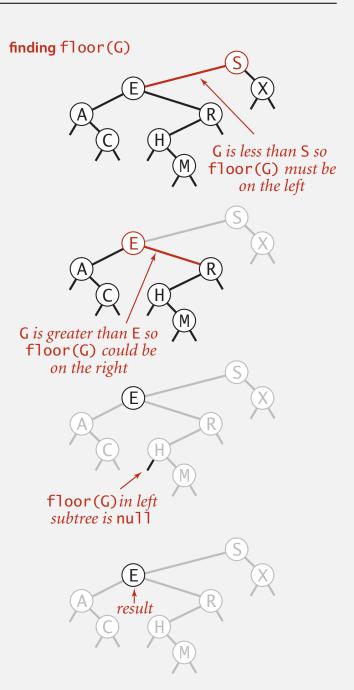
Q. How to find the floor / ceiling?

#### Computing the floor

Case 1. [k equals the key at root] The floor of k is k.

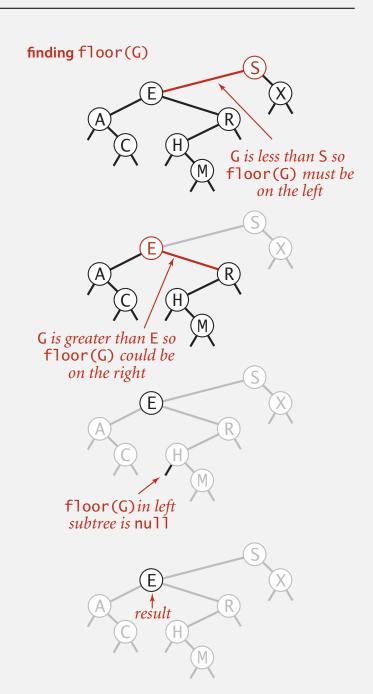
Case 2. [k is less than the key at root] The floor of k is in the left subtree.

Case 3. [k is greater than the key at root] The floor of k is in the right subtree (if there is any key  $\leq k$  in right subtree); otherwise it is the key in the root.



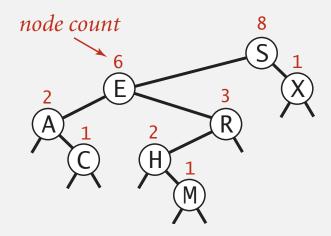
### Computing the floor

```
public Key floor(Key key)
  Node x = floor(root, key);
  if (x == null) return null;
   return x.key;
private Node floor(Node x, Key key)
  if (x == null) return null;
  int cmp = key.compareTo(x.key);
  if (cmp == 0) return x;
  if (cmp < 0) return floor(x.left, key);</pre>
  Node t = floor(x.right, key);
  if (t != null) return t;
  else
                  return x;
```



#### Subtree counts

In each node, we store the number of nodes in the subtree rooted at that node; to implement size(), return the count at the root.



Remark. This facilitates efficient implementation of rank() and select().

#### BST implementation: subtree counts

```
private class Node
{
   private Key key;
   private Value val;
   private Node left;
   private Node right;
   private int count;
}
```

```
public int size()
{ return size(root); }

private int size(Node x)
{
  if (x == null) return 0;
  return x.count;  ok to call
  when x is null
}
```

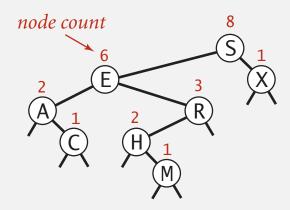
number of nodes in subtree

```
private Node put(Node x, Key key, Value val)
{
   if (x == null) return new Node(key, val, 1);
   int cmp = key.compareTo(x.key);
   if        (cmp < 0) x.left = put(x.left, key, val);
   else if (cmp > 0) x.right = put(x.right, key, val);
   else if (cmp == 0) x.val = val;
   x.count = 1 + size(x.left) + size(x.right);
   return x;
}
```

#### Rank

Rank. How many keys < k?

Easy recursive algorithm (3 cases!)



```
public int rank(Key key)
{  return rank(key, root); }

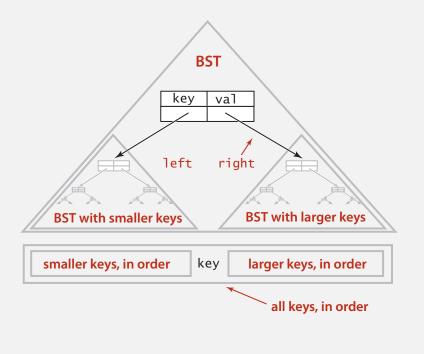
private int rank(Key key, Node x)
{
  if (x == null) return 0;
  int cmp = key.compareTo(x.key);
  if (cmp < 0) return rank(key, x.left);
  else if (cmp > 0) return 1 + size(x.left) + rank(key, x.right);
  else if (cmp == 0) return size(x.left);
}
```

#### Inorder traversal

- Traverse left subtree.
- Enqueue key.
- Traverse right subtree.

```
public Iterable<Key> keys()
{
    Queue<Key> q = new Queue<Key>();
    inorder(root, q);
    return q;
}

private void inorder(Node x, Queue<Key> q)
{
    if (x == null) return;
    inorder(x.left, q);
    q.enqueue(x.key);
    inorder(x.right, q);
}
```



Property. Inorder traversal of a BST yields keys in ascending order.

## BST: ordered symbol table operations summary

	sequential search	binary search	BST	
search	N	lg N	h	
insert	N	N	h	h = height of BST
min / max	N	1	h ←	(proportional to log N if keys inserted in random order)
floor / ceiling	N	lg N	h 🖊	
rank	N	lg N	h	
select	N	1	h	
ordered iteration	N log N	N	N	

order of growth of running time of ordered symbol table operations

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## ST implementations: summary

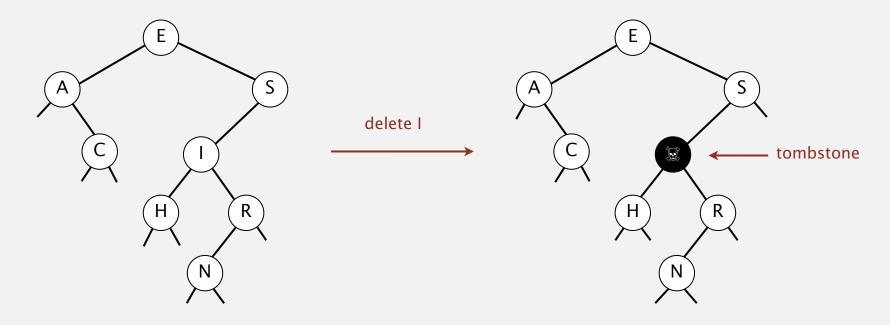
implementation	guarantee			average case			ordered	operations
	search	insert	delete	search hit	insert	delete	iteration?	on keys
sequential search (linked list)	N	N	N	N/2	N	N/2	no	equals()
binary search (ordered array)	lg N	N	N	lg N	N/2	N/2	yes	compareTo()
BST	N	N	N	1.39 lg N	1.39 lg N	???	yes	compareTo()

Next. Deletion in BSTs.

#### BST deletion: lazy approach

#### To remove a node with a given key:

- Set its value to null.
- Leave key in tree to guide search (but don't consider it equal in search).



Cost.  $\sim 2 \ln N'$  per insert, search, and delete (if keys in random order), where N' is the number of key-value pairs ever inserted in the BST.

Unsatisfactory solution. Tombstone (memory) overload.

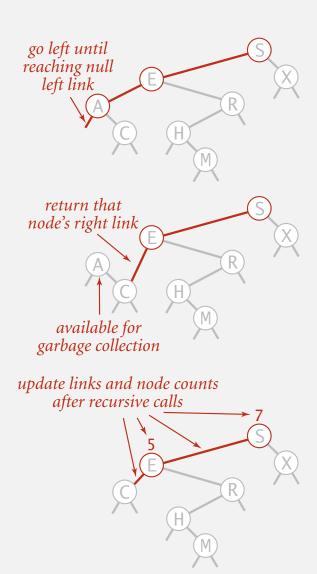
#### Deleting the minimum

#### To delete the minimum key:

- Go left until finding a node with a null left link.
- Replace that node by its right link.
- Update subtree counts.

```
public void deleteMin()
{    root = deleteMin(root); }

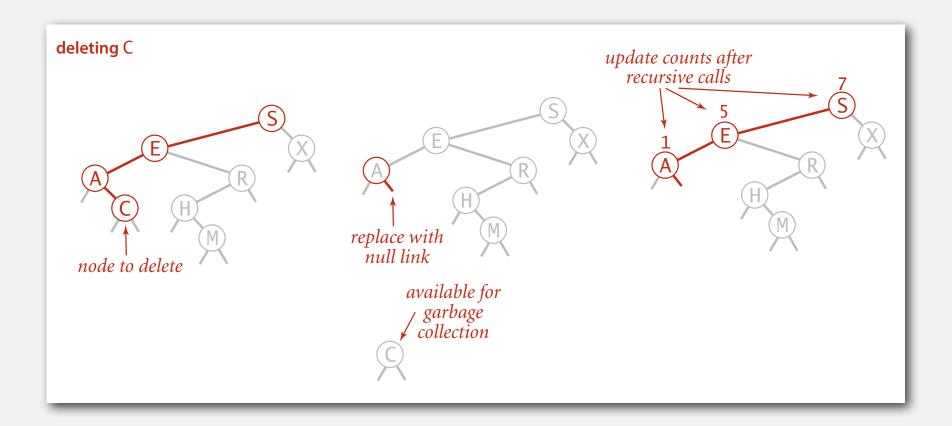
private Node deleteMin(Node x)
{
    if (x.left == null) return x.right;
    x.left = deleteMin(x.left);
    x.count = 1 + size(x.left) + size(x.right);
    return x;
}
```



#### Hibbard deletion

To delete a node with key k: search for node t containing key k.

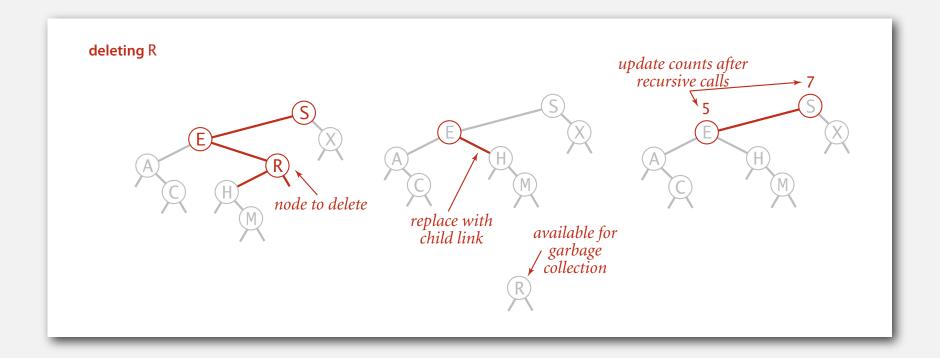
Case 0. [0 children] Delete t by setting parent link to null.



#### Hibbard deletion

To delete a node with key k: search for node t containing key k.

Case 1. [1 child] Delete t by replacing parent link.



#### Hibbard deletion

To delete a node with key k: search for node t containing key k.

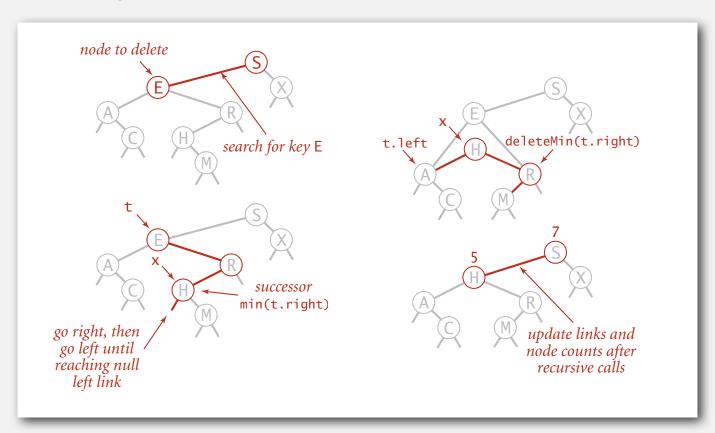
#### Case 2. [2 children]

Find successor x of t.

• Delete the minimum in t's right subtree.

• Put x in t's spot.

x has no left childbut don't garbage collect xstill a BST

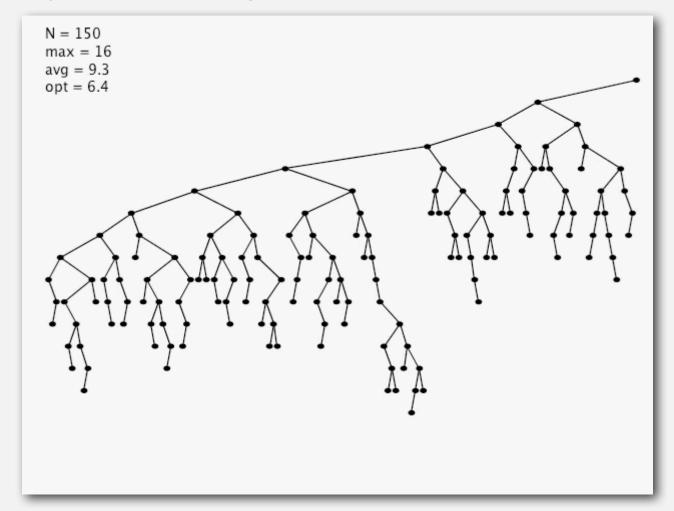


### Hibbard deletion: Java implementation

```
public void delete(Key key)
{ root = delete(root, key); }
private Node delete(Node x, Key key) {
   if (x == null) return null;
   int cmp = key.compareTo(x.key);
       (cmp < 0) x.left = delete(x.left, key); __</pre>
                                                                    search for key
   else if (cmp > 0) x.right = delete(x.right, key);
   else {
      if (x.right == null) return x.left;
                                                                     no right child
      if (x.left == null) return x.right;
                                                                     no left child
      Node t = x:
      x = min(t.right);
                                                                     replace with
                                                                      successor
      x.right = deleteMin(t.right);
      x.left = t.left;
   }
                                                                    update subtree
   x.count = size(x.left) + size(x.right) + 1;
                                                                       counts
   return x;
}
```

## Hibbard deletion: analysis

Unsatisfactory solution. Not symmetric.



Surprising consequence. Trees not random (!)  $\Rightarrow$  sqrt (N) per op. Longstanding open problem. Simple and efficient delete for BSTs.

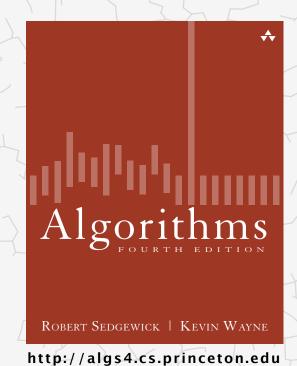
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other operations also become  $\sqrt{N}$  if deletions allowed

Next lecture. Guarantee logarithmic performance for all operations.

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